Macroeconomic Theory

Introduction to General Equilibrium Model

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1. Macroeconomic Analysis
   - Theory
   - Modelling
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   - Short and Long Run in the Economy
   - Static and Dynamic Economics
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   - Dynamic Analysis
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Macroeconomics I

Macroeconomics is concerned with the understanding of aggregate phenomena.

- Economic growth
- Business cycles
- Unemployment
- Inflation
- Exchange rate
- Interest rate (financial variables)
- Current account balance

Macroeconomics II

What is the objective of macroeconomic analysis:

- to understand these complex relations from a theoretical perspective
- to build models that help economists understand the complex relations,
- to explain aggregate behaviour in the economy.

Model I

- Theory is a set of structures.
- Model is an expression of theory.
- A model is a compact way of presenting a theory.
- Model needs an abstraction.
- A model is a lens.
- A model is a map.
- A model is window.

Model II

Models are economist's best laboratory.

What makes a good model?

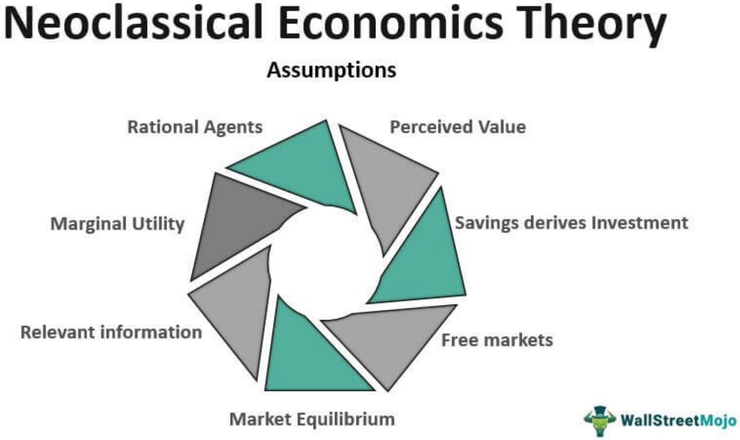
→ Logically consistent

→ Simple

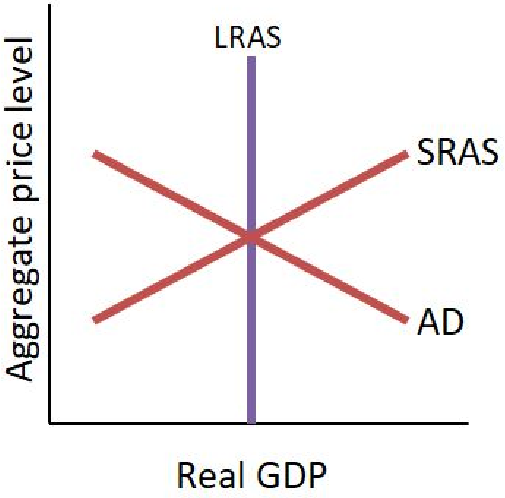
Neoclassical Economics I

- Also called: Welfare Economics, Walrasian economics, or Computable General Equilibrium Theory.
- Classical economics was centered around ethics and "moral philosophy". Adam Smith, David Ricardo, John Stuart Mill
- Neoclassical economics was built around Newtonian physics or "classical mechanics". Vilfredo Pareto, William Jevons, Leon Walras
- In the 1950s, Keynesian macroeconomic theories and neoclassical microeconomic theories were combined.
- The combination led to the neoclassical synthesis, which has dominated economic reasoning since then.

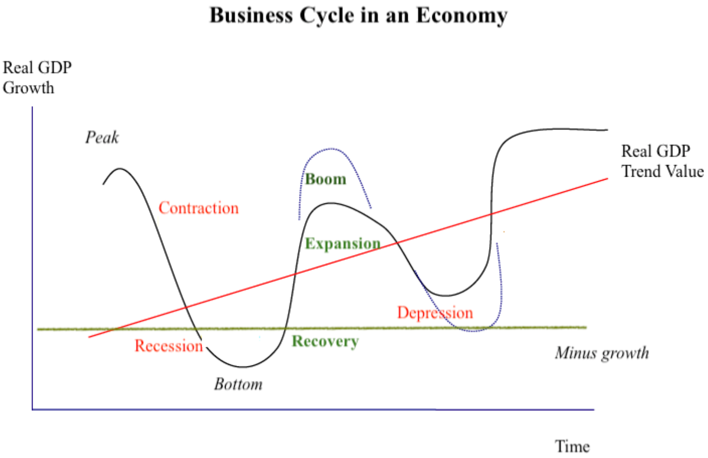
Neoclassical Economics II



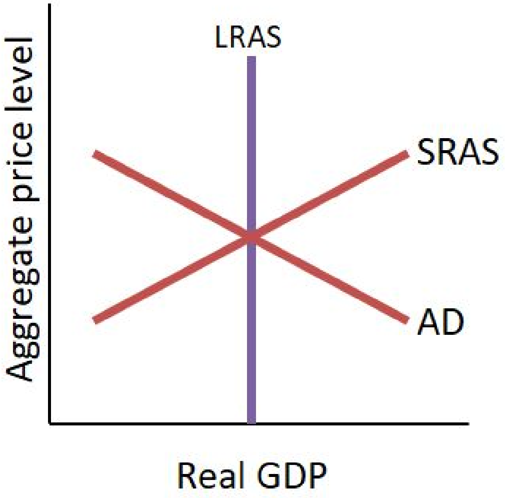
General Equilibrium



Business Cycles



Short and Long-run General Equilibrium



Short and Long Run

Short Run in the economy

Output is determined by demand.

Medium Run in the economy

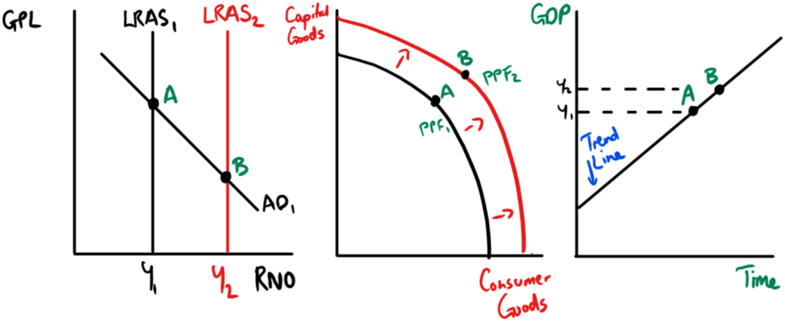
Output is determined by the level of technology, the capital stock and the labor force.

Long Run in the economy

Output is determined by education, research, savings and quality of governance.

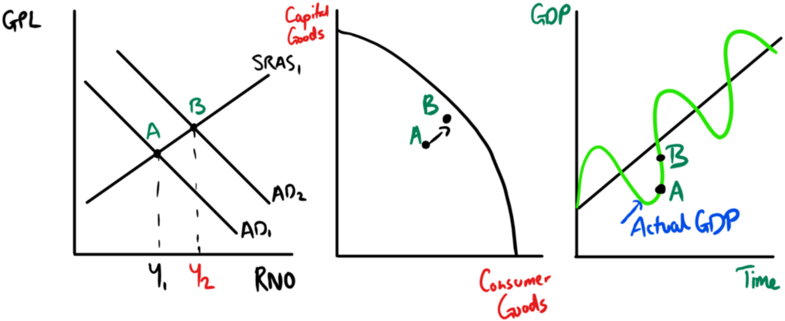
Long Run

Long-run Economic Growth

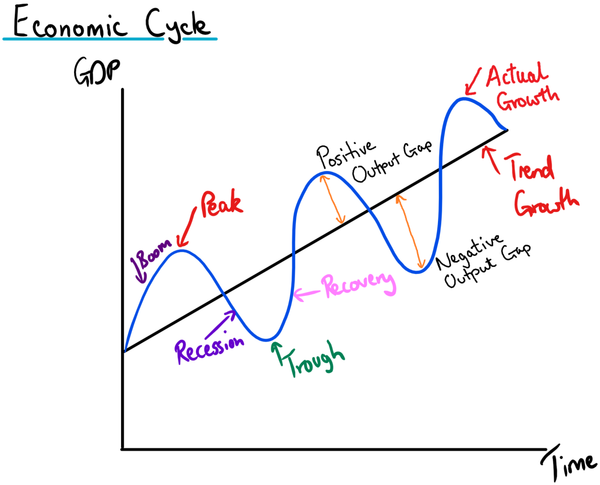


Short Run

Short-run Economic Growth



Economic Cycle



Differences I

1. Time
   - There is no time element in static economic analysis. All economic variables refer to the same point of time.
   - Static economy is also called a timeless economy.
   - Time element occupies an important role in dynamic economic analysis. Here all quantities must be dated.
   - Economic variables refer to the different points of time.

Differences II

2. Change

- Static analysis does not show the path of change. It only tells about the conditions of equilibrium.
- Dynamic economic analysis also shows the path of change.
- Static economics is called a "picture" whereas the dynamic economics is called a "movie" of the economy.

Differences III

3. Equilibrium

- Static economics studies only a particular point of equilibrium.
- Dynamic economics also studies the process by which equilibrium is achieved. As a result, there may be equilibrium or may be disequilibrium.
- Static analysis is a study of equilibrium only whereas dynamic analysis studies both equilibrium and disequilibrium.

Differences IV

4. Reality

- Static analysis is far from reality while dynamic analysis is nearer to reality.
- Static analysis is based on the unrealistic assumptions of perfect competition, perfect knowledge, etc.
- Here all the important economic variables like preferences, population, models of production, etc. are assumed to be constant.
- Dynamic analysis takes these economic variables as changeable.

Economic Dynamics

- Economic dynamics is a more realistic method of analysing the behaviour of the economy or certain economic variables through time.
- Dynamic Analysis or Economic Dynamics considers the relationship between relevant variables whose values belong to different points of time.
- The relations between certain variables, the values of which refer to the different points or different periods of time, are known as dynamic relationships.

Insights

- Economics is often interested in the behaviour of individuals or agents.
- These agents are assumed to behave rationally, that is, taking decisions that optimize their utility (in the case of households) or profits (in the case of firms)
- Optimization implies that agents maximize their utility/profits subject to the restrictions they face.
- Optimization for one period $\Rightarrow$ Static Optimization
- Optimization for more than one period $\Rightarrow$ Dynamic Optimization

The Kuhn-Tucker Theorem I

Consider a simple constrained optimization problem:

$x \in \mathbf{R}$ choice variable $F : \mathbf{R} \to \mathbf{R}$ objective function, continuously differentiable $c \geq G(x)$ constraint, with $c \in \mathbf{R}$ and $G : \mathbf{R} \to \mathbf{R}$, also continuously differentiable.

The problem can be stated as:

$$\max_x F(x) \text{ subject to } c \geq G(x)$$

The Kuhn-Tucker Theorem II

- This problem is "simple" because it is static and contains no random or stochastic elements.
- There is no uncertainty
- This problem is also "simple" because it has a single choice variable and a single constraint.
- The easiest way to solve this problem is via the method of Lagrange multipliers.
- We use Lagrange's Theorem or, in its most general form, the Kuhn-Tucker Theorem.

The Kuhn-Tucker Theorem III

To prove this theorem, begin by defining the Lagrangian:

$$L(x, \lambda) = F(x) + \lambda[c - G(x)]$$

for any $x \in \mathbf{R}$ and $\lambda \in \mathbf{R}$.

Suppose that x^* maximizes $F(x)$ subject to $c \geq G(x)$, where F and G are both continuously differentiable, and suppose that $G'(x^*) \neq 0$. Then there exists a value λ^* of λ such that x^* and λ^* satisfy the following four conditions:

$$L_1(x^*, \lambda^*) = F'(x^*) - \lambda^* G'(x^*) = 0$$

$$L_2(x^*, \lambda^*) = c - G(x^*) \geq 0$$

$$\lambda^* \geq 0$$

and

$$\lambda^* [c - G(x^*)] = 0$$

An Example for Utility Maximization I

A consumer has a utility function defined over consumption of two goods:
$U(c_1, c_2)$
Prices: p_1 and p_2
Income: I
Budget constraint: $I \geq p_1c_1 + p_2c_2 = G(c_1, c_2)$

The consumer's problem is:
$\max_{c_1,c_2} U(c_1, c_2)$ subject to $I \geq p_1c_1 + p_2c_2$

The Kuhn-Tucker theorem tells us that if we set up the Lagrangian:

$$L(c_1, c_2, \lambda) = U(c_1, c_2) + \lambda(I - p_1c_1 - p_2c_2)$$

An Example for Utility Maximization II

Then the optimal consumptions c_1^* and c_2^* and the associated multiplier λ^* must satisfy the FOC:

$$L_1\left(c_1^*, c_2^*, \lambda^*\right) = U_1\left(c_1^*, c_2^*\right) - \lambda^* p_1 = 0$$

and

$$L_2\left(c_1^*, c_2^*, \lambda^*\right) = U_2\left(c_1^*, c_2^*\right) - \lambda^* p_2 = 0$$

Move the terms with minus signs to the other side, and divide the first of these FOC by the second to obtain

$$\frac{U_1\left(c_1^*, c_2^*\right)}{U_2\left(c_1^*, c_2^*\right)} = \frac{p_1}{p_2}$$

which is just the familiar condition that says that the optimizing consumer should set the slope of his or her indifference curve, the marginal rate of substitution, equal to the slope of his or her budget constraint, the ratio of prices.

An Example for Utility Maximization III

Now consider I as one of the model's parameters, and let the functions $c_1^*(I), c_2^*(I)$, and $\lambda^*(I)$ describe how the optimal choices c_1^* and c_2^* and the associated value λ^* of the multiplier depend on I. In addition, define the maximum value function as

$$V(I) = \max_{c_1, c_2} U\left(c_1, c_2\right) \text{ subject to } I \geq p_1 c_1 + p_2 c_2$$

The Kuhn-Tucker theorem tells us that

$$\lambda^*(I)\left[I - p_1 c_1^*(I) - p_2 c_2^*(I)\right] = 0$$

An Example for Utility Maximization IV

and hence

$$V(I) = U\left[c_1^*(I), c_2^*(I)\right] = U\left[c_1^*(I), c_2^*(I)\right] + \lambda^*(I)\left[I - p_1 c_1^*(I) - p_2 c_2^*(I)\right]$$

The envelope theorem tells us that we can ignore the dependence of c_1^* and c_2^* on I in calculating

$$V'(I) = \lambda^*(I),$$

which gives us an interpretation of the multiplier λ^* as the marginal utility of income.

Basics for Dynamic Optimization I

- We need to index the variables that enter into the problem by t, in order to keep track of changes in those variables that occur over time.
- We need to distinguish between two types of variables
- Stock variables: stock of capital, assets, or wealth
- Flow variables: output, consumption, saving, or labor supply per unit of time.
- We need to introduce constraints that describe the evolution of stock variables over time.
- It means that larger flows of savings or investment today will lead to larger stocks of wealth or capital tomorrow.

Basics for Dynamic Optimization II

Consider a dynamic optimization in discrete time, that is, in which time can be indexed by $t = 0, 1, \ldots, T$.
$y_t =$ stock variable
$z_t =$ flow variable

Objective function:

$$\sum_{t=0}^{T} \beta^t F\left(y_t, z_t; t\right)$$

We can allow for a wider range of possibilities by letting the functions as well as the variables depend on the time index t.
Discount factor $\Rightarrow 0 < \beta \leq 1$

Basics for Dynamic Optimization III

Constraint describing the evolution of the stock variable:

$$Q(y_t, z_t; t) \geq y_{t+1} - y_t$$

or

$$y_t + Q(y_t, z_t; t) \geq y_{t+1}$$

for all $t = 0, 1, \ldots, T$

Constraint applying to variables within each period:

$$c \geq G(y_t, z_t; t)$$

for all $t = 0, 1, \ldots, T$

Constraints on initial and terminal values of stock:

$$y_0$$

$$y_{T+1} \geq y^*$$

Basics for Dynamic Optimization IV

The dynamic optimization problem can now be stated as:
choose sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ to maximize the objective function subject to all of the constraints.

Notes:
a) It is important for the application of the maximum principle that the problem be additively time separable: that is, the values of F, Q, and G at time t must depend on the values of y_t and z_t only at time t.
b) Although the constraints describing the evolution of the stock variable and applying to the variables within each period can each be written in the form of a single equation, it must be emphasized that these constraints must hold for all $t = 0, 1, \ldots, T$. That is, each of these equations actually describes $T + 1$ constraints.

The Maximum Principle I

Consider the discrete time dynamic optimization problem of choosing sequences $\{z_t\}_{t=0}^{T}$ and $\{y_t\}_{t=1}^{T+1}$ to maximize the objective function

$$\sum_{t=0}^{T} \beta^t F\left(y_t, z_t; t\right)$$

subject to the constraints

$$y_t + Q\left(y_t, z_t; t\right) \geq y_{t+1}$$

for all $t = 0, 1, \ldots, T$

$$c \geq G\left(y_t, z_t; t\right)$$

for all $t = 0, 1, \ldots, T$, y_0 given and

$$y_{T+1} \geq y^*.$$

The Maximum Principle II

Associated with this problem, define the maximized Hamiltonian

$$H\left(y_t, \pi_{t+1}; t\right) = \max_{z_t} \beta^t F\left(y_t, z_t; t\right) + \pi_{t+1} Q\left(y_t, z_t; t\right)$$

subject to $c \geq G\left(y_t, z_t; t\right)$.

Then the solution to the dynamic optimization problem must satisfy:
a) The first-order and complementary slackness conditions for the static optimization problem:

$$\beta^t F_z\left(y_t, z_t; t\right) + \pi_{t+1} Q_z\left(y_t, z_t; t\right) - \lambda_t G_z\left(y_t, z_t; t\right) = 0$$

and

$$\lambda_t\left[c - G\left(y_t, z_t; t\right)\right] = 0$$

for all $t = 0, 1, \ldots, T$.

The Maximum Principle III

b) The pair of difference equations:

$$\pi_{t+1} - \pi_t = -H_y\left(y_t, \pi_{t+1}; t\right)$$

for all $t = 1, 2, \ldots, T$ and

$$y_{t+1} - y_t = H_\pi\left(y_t, \pi_{t+1}; t\right)$$

for all $t = 0, 1, \ldots, T$, where the derivatives of H can be calculated using the envelope theorem.

c) The initial condition y_0 given

The Maximum Principle IV

d) The terminal, or transversality, condition

$$\pi_{T+1}\left(y_{T+1} - y^{*}\right) = 0$$

in the case where $T < \infty$ or

$$\lim_{T\to\infty} \pi_{T+1}\left(y_{T+1} - y^{*}\right) = 0$$

in the case where $T = \infty$.

Basics I

- Dynamic general equilibrium is the foundation of modern macroeconomic models.
- We first explore the concept of general equilibrium using a simple static model.
- Our simple economy involves two groups of economic agents: consumers and firms.

Basics II

- The representative household determines the behaviour of consumers.
- The representative firm, determines the behaviour of firms.
- The representative household supplies labour and capital to the representative firm.
- The representative firm uses these factor inputs to produce output and sells the output back to the household.
- We use this model to explore how changes in the level of technology affect the labour market and the capital market and how the two markets interact with each other.

The Model I

In general, the representative household maximises utility.

For now, we simply assume that the household supplies labour L^s and capital K^s inelastically to earn a wage income W and a capital rental income R.

Perfectly inelastic supply implies a vertical labour supply curve $(L^s = \bar{L}\ )$ and a vertical capital supply curve $\left(K^s = \bar{K}\right)$

$\bar{L}$ and $\bar{K}$ are exogenous parameters.

The Model II

We now consider the firm's optimisation problem.

The representative firm hires labour L and rents capital K from the household to produce output Y.

Cobb-Douglas production function for the firm:

$$Y = AK^{\alpha}L^{1-\alpha}$$

$\alpha \in (0,1)$ is the degree of capital intensity in production
A is the exogenous level of technology.

The profit function Π :

$$\Pi = PY - RK - WL$$

We consider a perfectly competitive product market, in which the firm takes the market price P as given .

The Model III

To maximise profit, we differentiate profit function with respect to K and L :

$$\frac{\partial \Pi}{\partial K} = P\frac{\partial Y}{\partial K} - R = 0$$
$$\frac{\partial \Pi}{\partial L} = P\frac{\partial Y}{\partial L} - W = 0$$

Rewriting above equations

$$\frac{\partial Y}{\partial K} = \alpha A K^{\alpha-1} L^{1-\alpha} = \frac{R}{P}$$
$$\frac{\partial Y}{\partial L} = (1-\alpha) A K^{\alpha} L^{-\alpha} = \frac{W}{P}$$

The result: A profit-maximising firm equate
- the marginal product of capital to the real rental price R/P,
- the marginal product of labour to the real wage rate W/P.

The Model IV

The demand for capital and labour is determined by the profit-maximising behaviour of firms.
For the demand curves for capital and labour, last equations can be re-expressed as

$$K^d = \left(\frac{\alpha A}{R/P}\right)^{1/(1-\alpha)} L$$

$$L^d = \left[\frac{(1-\alpha)A}{W/P}\right]^{1/\alpha} K$$

- capital demand K^d is decreasing in the real rental price R/P
- labour demand L^d is decreasing in the real wage rate W/P.
- an increase in the level of technology increases the demand for capital and labour.
- an increase in the level of labour increases the demand for capital.
- an increase in the level of capital increases the demand for labour.

Equilibrium I

Combining the demand and supply curves of capital (labour) yields the equilibrium level of the real rental price (real wage rate) as follows:

$$\frac{R}{P} = \alpha A \left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha}$$
$$\frac{W}{P} = (1-\alpha) A \left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}$$

Together with the production function

$$Y = A\bar{K}^{\alpha}\bar{L}^{1-\alpha}$$

Equilibrium II

The results:

1) an increase in the level of technology A raises the level of output Y, the real rental price R/P (by increasing capital demand) and the real wage rate W/P (by increasing labour demand). (see Figures 1.1 and 1.2.)

2) an increase in the level of capital $\bar{K}$ increases the level of output Y and the real wage rate W/P but decreases the real rental price R/P.

3) an increase in the level of labour $\bar{L}$ increases the level of output Y and the real rental price R/P but decreases the real wage rate W/P.

Equilibrium III

As we can see,

changes in the supply of one factor input (e.g., labour)
not only affect its own market (e.g., labour market)
but also affect the other market (e.g., capital market).

This interaction between the two markets represents a general-equilibrium effect.

Equilibrium IV

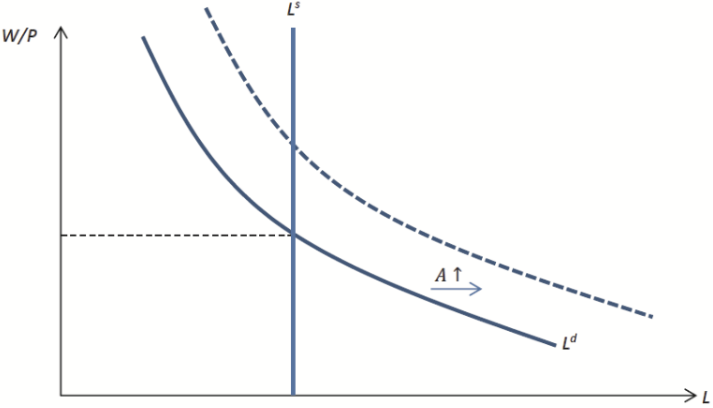


Figure 1.1. Labour market: Inelastic labour supply.

Equilibrium V

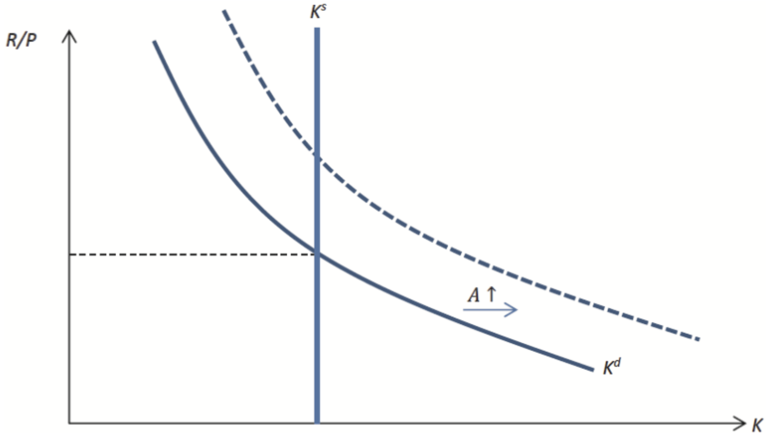


Figure 1.2. Capital market: Inelastic capital supply.

Elastic Labor Supply I

Let's generalise the model by allowing for elastic labour supply.

Then we have an upward-sloping labour supply curve in the labour market.

In this case, an increase in the level of technology A raises the real wage rate and the equilibrium level of labour by shifting the labour demand curve to the right; see Figure 1.3.

The resulting increase in the equilibrium level of labour in turn has a general-equilibrium effect on the capital market by shifting the capital demand curve further to the right and causing an additional positive effect on the real rental price; see Figure 1.4.

Elastic Labor Supply II

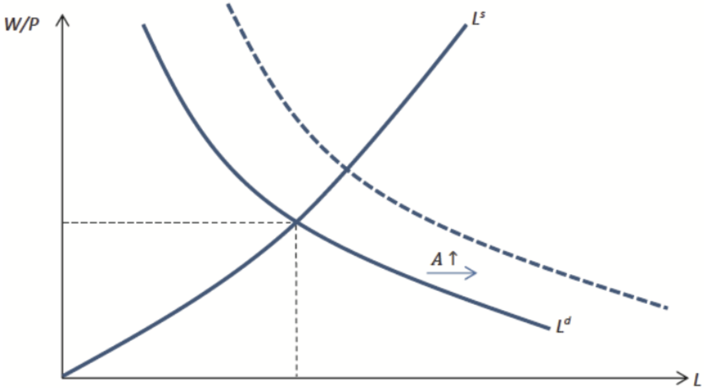


Figure 1.3. Labour market: Elastic labour supply.

Elastic Labor Supply III

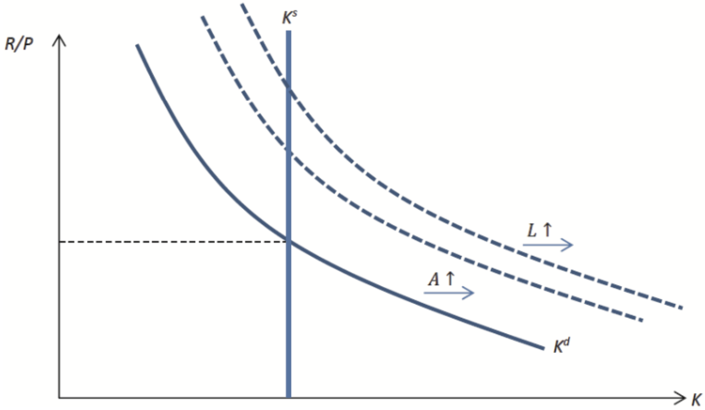


Figure 1.4. Capital market: Elastic labour supply.

Neutrality of Money I

So far, we have only determined the real wage rate W/P and the real rental price R/P.

However, we haven't determined the nominal wage rate W and the nominal rental price R, which in turn are determined by the price level P.

To determine the price level, we introduce the quantity theory of money:

$$MV = PY$$

M is the level of money supply and V is the velocity of money in the economy.
For simplicity, we set $V = 1$

Neutrality of Money II

Substituting Y, we have

$$P = \frac{M}{Y} = \frac{M}{A\bar{K}^{\alpha}\bar{L}^{1-\alpha}}$$

This equation shows that the level of money supply in the economy determines the price level.

When the central bank increases the level of money supply M, the price level P increases by the same proportion without affecting the level of output Y.

The real wage rate W/P and the real rental price R/P also remain unchanged, whereas the nominal wage rate W and the nominal rental price R increase by the same proportion as the level of money supply M.

Neutrality of Money III

To see this, we substitute price level equation into real rental price and real wage equations:

$$W = (1-\alpha)A\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} P = (1-\alpha)\frac{M}{\bar{L}}$$

$$R = \alpha A\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} P = \alpha\frac{M}{\bar{K}}$$

This neutrality of money would also hold when the supply of capital and/or labour is elastic.

The neutrality of money arises because the price level in the economy changes immediately to offset any change in the money supply.

Summary I

- We use a simple static general-equilibrium model to explore the effects of technology on the economy.
- We find that an increase in the level of technology raises the level of output, the real rental price and the real wage rate.
- In the case of elastic labour supply, the equilibrium level of labour also increases, which in turn has a general-equilibrium effect on the capital market by shifting the capital demand curve further to the right and causing a larger increase in the real rental price.

Summary II

- We also explore the effects of changes in the level of money supply and find that money supply only affects nominal variables (i.e., the price level, the nominal rental price and the nominal wage rate) without affecting any of the real variables (i.e., the level of output, the real rental price, the real wage rate, the level of labour and the level of capital).
- This result is known as the classical dichotomy, according to which real and nominal variables can be determined separately.

Intro I

- We convert the static general-equilibrium model into a dynamic general-equilibrium model.
- This model is the foundation of modern macroeconomics.
- The representative household chooses consumption and saving to maximize lifetime utility.
- To solve this dynamic optimization problem, we use a mathematical tool known as the Hamiltonian.
- This analysis enables us to endogenize the equilibrium levels of macroeconomic variables, such as capital and output, in order to explore their determinants, such as the level of technology and the preference of the representative household.

Household I

In the neoclassical growth model, the representative household has a utility function u_t at time t.

For simplicity, we consider a log utility function $u_t = \ln C_t$ that depends on consumption C_t.

$\Rightarrow$increasing consumption makes the household better off.

Furthermore, the log utility function has a number of nice properties.

- It features diminishing marginal utility.
- The $\log$ of zero is negative infinity so that the household would avoid zero consumption.

Household II

A forward-looking household should not only care about utility at time t but also lifetime utility, which is given by

$$U = u_0 + u_1 + u_2 + \cdots = \sum_{t=0}^{T} u_t$$

where T is the length of a lifetime.

Utility equation assumes that current utility and future utility carry the same weight, which is unrealistic because future consumption is often discounted.

Household III

To capture discounting, we introduce a discount rate $\rho > 0$ so that utility function becomes

$$U = u_0 + \frac{u_1}{1+\rho} + \frac{u_2}{(1+\rho)^2} + \cdots = \sum_{t=0}^{\infty} \frac{u_t}{(1+\rho)^t}$$

where we assume that a lifetime is long enough to be approximated by infinity.

Note: As t becomes very large, the discounting would make $\frac{u_t}{(1+\rho)^t}$ not to matter too much in the utility function U.

Household IV

In the rest of the analysis, we will use a mathematical tool known as Hamiltonian that solves dynamic optimisation problems in continuous time.

In intermediate microeconomics, students often use the Lagrangian for solving static constrained optimisation problems. In the case of dynamic optimisation problems, we use the Hamiltonian instead.

Therefore, we need to rewrite utility function in continuous time using the integral as

$$U = \int_0^\infty e^{-\rho t} u_t dt = \int_0^\infty e^{-\rho t} \ln C_t dt$$

where the continuous-time discount factor $e^{-\rho t}$ replaces the discrete time discount factor $(1+\rho)^{-t}$.

Household V

- The household inelastically supplies L units of labour to earn a wage income W_t.
- It accumulates capital K_t and rents it to the representative firm to earn a capital-rental income R_t.
- We assume that capital is the only productive asset in the economy.

If we normalize the price of output to unity, then the asset-accumulation equation:

$$\dot{K}_t = R_t K_t + W_t L - C_t$$

where $\dot{K}_t \equiv \partial K_t / \partial t$ is the change in the level of capital with respect to time t.

Here we have assumed a zero depreciation rate of capital (i.e., $\delta = 0$).

Note: In general, the asset-accumulation is given by

$\dot{K}_t = (R_t - \delta) K_t + W_t L - C_t$

Hamiltonian I

The household maximizes

$$U = \int_0^\infty e^{-\rho t} u_t dt = \int_0^\infty e^{-\rho t} \ln C_t dt$$

subject to

$$\dot{K}_t = R_t K_t + W_t L - C_t$$

To solve this dynamic optimization problem, we use the Hamiltonian. The Hamiltonian function H_t is given by

$$H_t = \ln C_t + \lambda_t \left(R_t K_t + W_t L - C_t \right).$$

The Hamiltonian function at time t consists of (a) the utility function $\ln C_t$, (b) the right-hand side of the asset-accumulation equation $R_t K_t + W_t L - C_t$, and (c) a multiplier λ_t for the asset-accumulation equation.

Hamiltonian II

To maximise the household's utility, we derive the first-order conditions:

$$\frac{\partial H_t}{\partial C_t} = \frac{1}{C_t} - \lambda_t = 0$$
$$\frac{\partial H_t}{\partial K_t} = \lambda_t R_t = \lambda_t \rho - \dot{\lambda}_t$$

Note that K_t is a state variable (i.e., a variable that accumulates over time), so we have to treat its first-order condition differently.
Instead of equating $\partial H_t / \partial K_t$ to zero, we set $\partial H_t / \partial K_t = \lambda_t \rho - \dot{\lambda}_t$
Taking the log of FOC-1 yields

$$\ln C_t = -\ln \lambda_t.$$

Hamiltonian III

Differentiating both sides of the equation with respect to t yields

$$\frac{\dot{C}_t}{C_t} = -\frac{\dot{\lambda}_t}{\lambda_t}$$

Note that $\frac{\partial \ln C_t}{\partial t} = \frac{1}{C_t}\frac{\partial C_t}{\partial t} = \frac{\dot{C}_t}{C_t}$.

Substituting this equation into FOC-2 yields

$$\frac{\dot{C}_t}{C_t} = -\frac{\dot{\lambda}_t}{\lambda_t} = R_t - \rho$$

which is the optimal path of consumption chosen by the household.

Hamiltonian IV

The optimal consumption path states that when the rental price R_t is greater than the discount rate ρ, the household's consumption should be increasing over time $\left(\text{i.e., } \dot{C}_t > 0\right)$.
Intuitively, when the return to capital is high relative to the household's discount rate, the household should decrease current consumption and increase saving in order to invest in capital.
As a result, consumption is increasing over time.

Conversely, when the rental price R_t is less than the discount rate ρ, the household's consumption should be decreasing over time $\left(\text{i.e., } \dot{C}_t < 0\right)$.
Intuitively, when the return to capital is low relative to the household's discount rate, the household should increase current consumption and decrease investment in capital.
As a result, consumption is decreasing over time.

Firm I

To derive the equilibrium of the economy, we also need to consider the firm's optimization problem.

There is a representative firm in the economy, and this firm hires labour L_t and rents capital K_t from the household to produce output Y_t.

Cobb-Douglas production function for the firm:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha},$$

where the parameter $\alpha \in (0, 1)$ is the degree of capital intensity in production and A is the exogenous level of technology.

The profit function: $\Pi_t = Y_t - R_t K_t - W_t L_t$

Firm II

Recall that we have chosen Y_t as the numeraire (i.e., the price of Y_t is normalised to unity).

Differentiating profit function with respect to K_t and L_t yields

$$\frac{\partial \Pi_t}{\partial K_t} = \frac{\partial Y_t}{\partial K_t} - R_t = \alpha A K_t^{\alpha-1} L_t^{1-\alpha} - R_t = 0$$
$$\frac{\partial \Pi_t}{\partial L_t} = \frac{\partial Y_t}{\partial L_t} - W_t = (1-\alpha) A K_t^{\alpha} L_t^{-\alpha} - W_t = 0$$

These two equations are the demand functions for K_t and L_t.

Steady-State Equilibrium I

Substituting $R_t = \alpha A K_t^{\alpha-1} L_t^{1-\alpha}$ from FOC-1 into optimal path of consumption yields

$$\frac{\dot{C}_t}{C_t} = \underbrace{\alpha A K_t^{\alpha-1} L^{1-\alpha}}_{=MPK_t} - \rho$$

where we have set $L_t = L$.

This equation shows that the optimal path of consumption is determined by the return to capital, which in turn is determined by the marginal product of capital MPK_t.

Substituting $R_t = \alpha A K_t^{\alpha-1} L^{1-\alpha}$ and $W_t = (1-\alpha) A K_t^{\alpha} L^{-\alpha}$ from FOC's into asset-accumulation equation yields the capital-accumulation equation:

$$\dot{K}_t = \alpha A K_t^{\alpha} L^{1-\alpha} + (1-\alpha) A K_t^{\alpha} L^{1-\alpha} - C_t = A K_t^{\alpha} L^{1-\alpha} - C_t.$$

Steady-State Equilibrium II

$$\dot{K}_t = \alpha A K_t^{\alpha} L^{1-\alpha} + (1-\alpha) A K_t^{\alpha} L^{1-\alpha} - C_t = A K_t^{\alpha} L^{1-\alpha} - C_t.$$

This equation shows that the accumulation of capital is determined by capital investment, which is the difference between output and consumption.

Equations are two differential equations in C_t and K_t, and these two equations determine the behaviour of the economy.

Now we solve for the steady-state equilibrium. In the steady state, all variables are constant, such that $\dot{C}_t = 0$ and $\dot{K}_t = 0$. Imposing $\dot{C}_t = 0$ on the optimal consumption path yields the steady-state equilibrium level of capital:

$$K^* = \left(\frac{\alpha A}{\rho}\right)^{1/(1-\alpha)} L$$

Steady-State Equilibrium III

$$K^* = \left(\frac{\alpha A}{\rho}\right)^{1/(1-\alpha)} L$$

which is increasing in the level of technology A and decreasing in the discount rate ρ.

Intuitively, a higher level of technology A increases the return to capital and encourages the household to accumulate more capital.

In contrast, a higher discount rate makes future consumption less attractive to the household, which prefers current consumption and accumulates less capital.

$\Rightarrow$ This equation also allows us to quantify the effects on K^*.

Steady-State Equilibrium IV

$$K^* = \left(\frac{\alpha A}{\rho}\right)^{1/(1-\alpha)} L$$

For example, the elasticity of K^* with respect to A is $1/(1-\alpha)$.

To see this, $\partial \ln K^* / \partial \ln A = 1/(1-\alpha)$

Note that $\frac{d \ln K^*}{d \ln A} = \frac{dK^*}{K^*} / \frac{dA}{A}$, which is the elasticity of K^* with respect to A.

In other words, increasing technology A by 1% would lead to an increase in capital K^* by $1/(1-\alpha)$ percent.
If $\alpha = 1/3$, then $1/(1-\alpha)$ is 1.5.
If $\alpha = 1/2$, then $1/(1-\alpha)$ is 2 .

Steady-State Equilibrium V

Using the production function $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, we can derive the steady-state equilibrium level of output:

$$Y^* = A\left(K^*\right)^{\alpha} L^{1-\alpha} = \left(\frac{\alpha A}{\rho}\right)^{\alpha/(1-\alpha)} AL$$

which is increasing in A via a direct effect from A on Y and an indirect effect from A on K^*.

The elasticity of Y^* with respect to A is also $1/(1-\alpha)$. Note that $1 + \alpha/(1-\alpha) = 1/(1-\alpha)$

Steady-state output Y^* is decreasing in ρ because a smaller capital stock K^* reduces the level of output Y^*.

Steady-State Equilibrium VI

Imposing $\dot{K}_t = 0$ on the capital-accumulation equation

$$\dot{K}_t = \alpha A K_t^{\alpha} L^{1-\alpha} + (1-\alpha) A K_t^{\alpha} L^{1-\alpha} - C_t = A K_t^{\alpha} L^{1-\alpha} - C_t.$$

yields the steady-state equilibrium level of consumption:

$$C^* = A \left(K^*\right)^{\alpha} L^{1-\alpha} = \left(\frac{\alpha A}{\rho}\right)^{\alpha/(1-\alpha)} AL$$

which is also increasing in A and decreasing in ρ because $C^* = Y^*$.

In this special case of $\delta = 0$, the steady-state equilibrium level of investment is $I^* = Y^* - C^* = 0$. However, in the more general case of $\delta > 0$, the steady-state equilibrium level of investment would be positive

Summary I

- We explore the concept of dynamic general equilibrium by developing the neoclassical growth model.
- The model features a utility-maximising representative household, which chooses consumption and saving optimally.
- We use the Hamiltonian to solve this dynamic optimisation problem and derive the household's optimal consumption path,
- A profit-maximising representative firm interacts with the utility-maximising household in the market economy that determines the allocation of resources in equilibrium.
- Then, we derive the steady-state equilibrium levels of capital and output, which are both increasing in the level of technology but decreasing in the household's discount rate and also the depreciation rate of capital.